

# Asymptotics for the maximum of a modulated random walk with heavy-tailed increments

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## Abstract

We consider asymptotics for the maximum of a modulated random walk whose increments  $\xi_n^{X_n}$  are heavy-tailed. Of particular interest is the case where the modulating process  $X$  is regenerative. Here we study also the maximum of the recursion given by  $W_0 = 0$  and, for  $n \geq 1$ ,  $W_n = \max(0, W_{n-1} + \xi_n^{X_n})$ .

## 1 Introduction

Let  $S_n = \sum_{i=1}^n \xi_i$  be a sum of i.i.d. random variables (r.v.s) with a negative finite mean  $\mathbf{E}\xi_1 = -a < 0$ . The common distribution of the random variables  $\xi_n$  is assumed to be right heavy-tailed (i.e.  $\mathbf{E} \exp(\lambda \xi_1) = \infty$  for all  $\lambda > 0$ ). Moreover, the second tail of this distribution is assumed to be subexponential (see Section 2 for definitions). Then the classical result (see, e.g., [12]) states that, as  $y \rightarrow \infty$ ,

$$\mathbf{P}(\sup_n S_n > y) \sim \frac{1}{a} \int_y^\infty \mathbf{P}(\xi_1 > t) dt. \quad (1)$$

We consider here a more general random walk

$$S_n = \sum_{i=1}^n \xi_i^{X_i} \quad (2)$$

whose increments  $\xi_n^{X_n}$  are modulated by an independent sequence  $X = \{X_n\}_{n \geq 1}$  (see Section 2 for more precise definitions and notation). We assume that  $S_n \rightarrow -\infty$  a.s. and find conditions which are sufficient for the probability of the “rare” event  $\mathbf{P}(\sup_n S_n > y)$  to behave asymptotically (as  $y \rightarrow \infty$ ) similarly to (1). The results obtained may be applied to the study of complex stochastic models with modulated input.

Particular cases, with  $X$  a finite Markov chain, were considered in [2] and [1]. S. Asmussen ([4]) proposed an approach for getting the asymptotics for  $\mathbf{P}(\sup_n S_n > y)$  on the basis of a regenerative structure: if the maximum of the partial sums over a typical cycle behaves asymptotically as the end-to-end sum, and these asymptotics are subexponential, then

the result (1) stays the same. In [5], the authors assumed that  $X$  is countably-valued, a certain dependence between the  $X_n$  and the  $\xi_n^x$  was allowed, and some homogeneity in  $x$  of the distributions of the random variables  $\xi_n^x$  was required. By the use of matrix-analytic methods, they found the asymptotics for the stationary distribution of a Markov chain with increments  $\xi_n^{X_n}$ .

In [7], upper and lower bounds were found for the asymptotics of  $\mathbf{P}(R > y)$ , as  $y \rightarrow \infty$ , where  $R$  is the stationary response time in a tandem queue. Then, in [9], the asymptotics for the stationary waiting time  $W$  in the second queue were studied. Note that  $W$  may be represented as the limit of a recursion

$$W_n = \max(0, W_{n-1} + \xi_n^{X_n})$$

where  $X = \{X_n\}$  forms a Harris ergodic Markov chain. In [6], the exact asymptotics for  $\mathbf{P}(R > y)$  were found. The proof is based on ideas similar to that of Lemma 2 of the present paper.

Finally, nice overviews on the current state of large deviations theory in the presence of heavy tails were given in [10] and in recent new books [8] and [3].

We state our main results in Section 2. We consider in particular the case where the modulating process  $X$  is regenerative, where we give also an instructive example and counterexample. The latter shows our conditions on the tail of the distribution of the regeneration time to be best possible—in a sense made clear there. We study also the queueing theory recursion given by  $W_0 = 0$  and, for  $n \geq 1$ ,  $W_n = \max(0, W_{n-1} + \xi_n^{X_n})$ .

In Section 3 we collect together some useful known results, most of which are required for our proofs. These are given in Section 4. Perhaps the key result of the entire paper is Lemma 2 of that section, which develops an idea found also in [6].

## 2 The main results

Let  $(\mathcal{X}, \mathcal{B})$  be a measurable space and  $X = \{X_n\}_{n \geq 1}$  an  $\mathcal{X}$ -valued discrete-time random process. Let  $P : \mathcal{X} \times \mathcal{B}_0 \rightarrow [0, 1]$  (where  $\mathcal{B}_0$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$ ) be a function such that

- (i) for every  $x \in \mathcal{X}$ ,  $P(x, \cdot)$  is a probability measure;
- (ii) for every  $B \in \mathcal{B}_0$ ,  $P(\cdot, B)$  is a measurable function.

For each  $x \in \mathcal{X}$ , let  $F_x$  denote the distribution function of  $P(x, \cdot)$ . For each integer  $n \geq 1$ , introduce the family of real-valued random variables  $\{\xi_n^x\}_{x \in \mathcal{X}}$ . Assume that these families are mutually independent (in  $n$ ), do not depend on the process  $X$ , and that, for each  $x \in \mathcal{X}$  and each  $n$ ,  $\xi_n^x$  has distribution function  $F_x$ . We define the *random walk*  $\{S_n\}_{n \geq 0}$  *modulated by the process*  $X$  by  $S_0 = 0$  and, for any  $n = 1, 2, \dots$ ,

$$S_n = \sum_{i=1}^n \xi_i^{X_i}.$$

Define also, for  $n \geq 1$ ,

$$M_n = \max_{0 \leq i \leq n} S_i,$$

and let

$$M = \sup_{n \geq 0} S_n.$$

Further, for each  $y > 0$ , define

$$\mu(y) = \min\{n \geq 1 : S_n > y\}.$$

Note that  $\mu(y) = \infty$  if and only if  $M \leq y$ .

We are interested the asymptotics of the upper-tail distribution of  $M$  under conditions which guarantee that the random walk  $S_n$  behaves sufficiently regularly and has a strictly negative drift, and where additionally the distribution functions  $F_x$  have, in some appropriate sense, heavy positive tails. More precisely, we wish to make statements, under such conditions, about the behaviour, for any  $B \in \mathcal{B}$  and as  $y \rightarrow \infty$ , of  $\mathbf{P}(M > y, X_{\mu(y)} \in B)$ .

Motivated by queueing theory applications, we are also interested in the behaviour of the process  $\{W_n\}_{n \geq 0}$  defined recursively by  $W_0 = 0$  and, for  $n \geq 1$ ,

$$W_n = \max(0, W_{n-1} + \xi_n^{X_n}). \quad (3)$$

We assume throughout that  $P$  is such that each probability measure  $P(x, \cdot)$  (i.e. each distribution  $F_x$ ) has a finite mean. We further assume throughout that there exist a distribution function  $F$  on  $\mathbb{R}_+$  with finite mean, and a measurable function  $c : \mathcal{X} \rightarrow \mathbb{R}_+$  such that

$$\overline{F}_x(y) \sim c(x)\overline{F}(y) \quad \text{as } y \rightarrow \infty, \quad \text{for all } x \in \mathcal{X}, \quad (4)$$

$$\sup_x \sup_{y \geq 0} \frac{\overline{F}_x(y)}{\overline{F}(y)} = L, \quad \text{for some } L < \infty. \quad (5)$$

Here, for any distribution function  $H$  on  $\mathbb{R}$ ,  $\overline{H}$  denotes the tail distribution given by  $\overline{H}(y) = 1 - H(y)$ .

The following two conditions on the pair  $(X, P)$  will be satisfied in all our results, either by hypothesis or as a consequence of more fundamental modelling assumptions. (We show below that these conditions may arise naturally in the case where the process  $X$  is regenerative, but they may also arise in other contexts.)

(C1) There exists some probability distribution  $\pi$  on  $(\mathcal{X}, \mathcal{B})$  such that, for some positive integer  $d$ ,

$$\frac{\mathbf{P}(X_n \in \cdot) + \dots + \mathbf{P}(X_{n+d-1} \in \cdot)}{d} \rightarrow \pi(\cdot), \quad \text{as } n \rightarrow \infty, \quad (6)$$

in total variation norm. Here define also

$$C(B) = \int_B c(x)\pi(dx), \quad B \in \mathcal{B}, \quad (7)$$

and put  $C = C(\mathcal{X})$ .

(C2) The pair  $(X, P)$  is such that

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = -a, \quad \text{a.s., for some } a > 0. \quad (8)$$

We need to recall the following definitions. For any distribution function  $H$  on  $\mathbb{R}$ , the integrated, or second-tail, distribution  $\overline{H}^s$  is given by

$$\overline{H}^s(y) = \min \left( 1, \int_y^\infty \overline{H}(t) dt \right).$$

A distribution function  $H$  on  $\mathbb{R}_+$  is *long-tailed* if  $\overline{H}(y) > 0$  for all  $y$  and, for any fixed  $z > 0$ ,

$$\frac{\overline{H}(y+z)}{\overline{H}(y)} \rightarrow 1 \quad \text{as } y \rightarrow \infty.$$

A distribution function  $H$  on  $\mathbb{R}_+$  is *subexponential* if  $\overline{H}(y) > 0$  for all  $y$  and

$$\frac{\overline{H}^{*2}(y)}{\overline{H}(y)} \rightarrow 2 \quad \text{as } y \rightarrow \infty.$$

(where  $H^{*2}$  is the convolution of  $H$  with itself). It is well known that any subexponential distribution is long-tailed.

We now have Theorems 1 and 2 below.

**Theorem 1.** *Assume that the conditions (C1) and (C2) hold and that  $F^s$  is long-tailed. Then*

$$\liminf_{y \rightarrow \infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} \geq \frac{C(B)}{a}, \quad \text{for all } B \in \mathcal{B}. \quad (9)$$

**Theorem 2.** *Assume that the conditions (C1) and (C2) hold, that  $F^s$  is subexponential, and that there exists a distribution function  $G$  with negative mean*

$$m(G) \equiv \int_{-\infty}^\infty t dG(t) < 0 \quad (10)$$

*such that  $\overline{F}_x(y) \leq \overline{G}(y)$  for all  $x$  and  $y$ . Then*

$$\lim_{y \rightarrow \infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} = \frac{C(B)}{a}, \quad \text{for all } B \in \mathcal{B}. \quad (11)$$

*Remark 1.* In an obvious sense, the best candidate for the distribution function  $G$  in Theorem 2 is the right-continuous version of  $\tilde{G}(y) = \sup_x F_x(y)$ .

As mentioned above, the conditions (C1) and (C2) are frequently satisfied in applications. Perhaps the most common instance occurs in the case where  $X$  is a *regenerative process*. Here there exists an increasing integer-valued random sequence  $0 \leq T_0 < T_1 < T_2 < \dots$  a.s., such that, if  $\tau_0 = T_0$ ,  $\tau_n = T_n - T_{n-1}$ ,  $n \geq 1$ , then

$$\begin{aligned} Z_0 &= \{\tau_0; X_1, \dots, X_{T_0}\}, \\ Z_n &= \{\tau_n; X_{T_{n-1}+1}, \dots, X_{T_n}\}, \quad n \geq 1, \end{aligned}$$

are mutually independent for  $n \geq 0$  and identically distributed for  $n \geq 1$ . Here  $\tau_0$  is the length of the 0th cycle,  $\tau \equiv \tau_1$  the length of the first cycle, etc; in particular, if  $\tau_0 = 0$ , then the 0th cycle is empty.

Assume also, for  $X$  regenerative as above, that  $\mathbf{E}\tau$  is finite. Then the condition (C1) is automatically satisfied with

$$d = \text{GCD}\{n : \mathbf{P}(\tau = n) > 0\}$$

and the probability measure  $\pi$  given by

$$\pi(B) = \mathbf{E} \left( \sum_{i=T_0+1}^{T_1} \mathbf{I}(X_i \in B) \right), \quad B \in \mathcal{B}, \quad (12)$$

where  $\mathbf{I}$  is the indicator function. Suppose also that the modulated random walk  $S_n$  is constructed as above, that

$$\int_{\mathcal{X}} \mathbf{E}(|\xi_1^x|) \pi(dx) < \infty, \quad (13)$$

and that

$$\int_{\mathcal{X}} \mathbf{E} \xi_1^x \pi(dx) = -a, \quad \text{for some } a > 0. \quad (14)$$

Then, it is an elementary exercise, using the Strong Law of Large Numbers, to show that the condition (C2) is satisfied with  $a$  as given by (14).

*Example 1.* For a particular example of such a random walk modulated by a regenerative process, consider a stable tandem queue  $GI/GI/1 \rightarrow GI/1$  which is defined by three mutually independent sequences  $\{t_n\}$ ,  $\{\sigma_n^{(1)}\}$ , and  $\{\sigma_n^{(2)}\}$  of i.i.d. random variables with  $\mathbf{E}t_1 > \max\{\mathbf{E}\sigma_1^{(1)}, \mathbf{E}\sigma_1^{(2)}\}$ . Here the  $t_n$  are the inter-arrival times at the first queue, while the  $\sigma_n^{(1)}$  and the  $\sigma_n^{(2)}$  are the service times at the first and second queues respectively. Let  $\{X_n\}$  be the sequence of inter-departure times from the first queue. Then this sequence is regenerative (the regeneration indices corresponding to those customers who arrive to find the first queue empty), and the distribution of  $X_n$  converges in the total variation norm to a stationary distribution with mean  $\mathbf{E}t_1$ . Consider the sequence  $\xi_n^{X_n} = \sigma_n^{(2)} - X_n$ . Under natural conditions (see Theorems 3–5 below), we can show that the tail distribution of the supremum of a modulated random walk with increments  $\xi_n^{X_n}$  asymptotically coincides with that of the stationary waiting time in the second queue.

In the case of a regenerative process as above we obtain, in Theorems 3 and 4 below, the conclusion (11) of Theorem 2 under weaker conditions than that given by (10). In each case the cost is that of suitable conditions on the distributions of the cycle times  $\tau_0$  and  $\tau$ . Both the theorems are adapted to typical queueing theory applications.

**Theorem 3.** *Assume that  $X$  is regenerative with  $\mathbf{E}\tau < \infty$ , that the conditions (13) and (14) hold (with  $\pi$  as given by (12)), and that  $F^s$  is subexponential. Assume also that*

$$\mathbf{P}(b\tau_0 > y) = o(\overline{F}^s(y)), \quad \mathbf{P}(b\tau > y) = o(\overline{F}(y)), \quad \text{as } y \rightarrow \infty, \quad (15)$$

*for all  $b > 0$ . Then the conclusion (11) of Theorem 2 again follows.*

*Remark 2.* As already discussed, the assumptions of Theorem 3 ensure that the earlier conditions (C1) and (C2) are satisfied.

*Remark 3.* It will follow from the proof of Theorem 3 that it is enough to assume that the condition (15) holds for a certain sufficiently large  $b$ .

*Remark 4.* The condition (15) holds for all  $b > 0$  if there exists some  $\lambda > 0$  such that both  $\mathbf{E}\exp(\lambda\tau)$  and  $\mathbf{E}\exp(\lambda\tau_0)$  are finite.

Under the conditions of Theorem 4 we relax the requirement that the condition (15) hold for all  $b > 0$ . This requirement may fail to be satisfied in some examples where the conditions of Theorem 4 are, however, quite natural—see, e.g., Example 1.

**Theorem 4.** *Assume that  $X$  is regenerative with  $\mathbf{E}\tau < \infty$ , that the conditions (13) and (14) hold, and that  $F^s$  is subexponential. Assume also that there exists a family  $\{G_x\}_{x \in \mathcal{X}}$  of distribution functions on  $\mathbb{R}_+$  such that  $G_x(y)$  is measurable in  $x$  for all  $y$ ,*

$$\overline{F}_x(y) \leq \overline{G}_x(y) \quad \text{for all } x \text{ and for all } y, \quad (16)$$

*and, for each  $x$ ,  $G_x$  may be represented as a distribution function of a difference of two independent r.v.s*

$$G_x(y) = \mathbf{P}(\zeta - b^x \leq y), \quad (17)$$

*where the distribution of  $\zeta$  does not depend on  $x$ ,*

$$\limsup_{y \rightarrow \infty} \frac{\mathbf{P}(\zeta > y)}{\overline{F}(y)} < \infty, \quad (18)$$

*$b^x$  is non-negative a.s., and*

$$\mathbf{E}_\pi b^X \equiv \int \mathbf{E}b^x \pi(dx) > \mathbf{E}\zeta. \quad (19)$$

*Assume further that the condition (15) holds for some  $b > \mathbf{E}\zeta$ . Then the conclusion (11) of Theorem 2 follows once more.*

*Remark 5.* The assumptions on  $X$  in Theorem 4 again ensure that the earlier condition (C1) is satisfied, while it follows from (16), (17) and (19) that the condition (14), and so the condition (C2), is satisfied.

*Remark 6.* It is easy to see that the asymptotics for  $\mathbf{P}(M > y)$  may be quite different from (11) if the assumption (15) fails. For instance, assume that the remaining conditions of Theorem 4 hold with  $\mathcal{X} = \mathcal{R}$ ,  $F_x = G_x$  for all  $x$ ,  $\zeta \geq 1$  a.s., and  $b^x \equiv 0$  for all  $x \neq 0$ . Assume also that  $X_0 = 0$ ,  $T_0 = 0$ ,  $\tau \equiv \tau_1 = \min\{n > 0: X_n = 0\}$ . The condition (19) here becomes  $\mathbf{E}b^0 > \mathbf{E}\zeta\mathbf{E}\tau$ . Then

$$M \geq \max(\tau_1 - 1, \tau_1 - b_{\tau_1}^0 + \tau_2 - 1, \dots) \equiv M^*.$$

If the second tail for  $\mathbf{P}(\tau > t)$  is subexponential (and so also long-tailed), then, by (1),

$$\mathbf{P}(M^* > y) \sim \frac{1}{\mathbf{E}(b^0 - \tau)} \int_y^\infty \mathbf{P}(\tau > t) dt.$$

Finally, again in the case where  $X$  is regenerative, we consider the process  $\{W_n\}_{n \geq 0}$  defined earlier by (3). In the special case where  $F_x = F$  for all  $x \in \mathcal{X}$  (so that  $\{\xi_n\}$  is an i.i.d. sequence), it is well known that the distributions of  $W_n$  and  $S_n$  coincide. However, this is not generally the case in the present setting.

**Theorem 5.** *Assume that  $X$  is regenerative with  $\mathbf{E}\tau < \infty$ . Then, under the conditions of either Theorem 2, Theorem 3 or Theorem 4,*

$$\lim_{y \rightarrow \infty} \frac{1}{\overline{F}^s(y)} \limsup_{n \rightarrow \infty} \mathbf{P}(W_n > y) = \lim_{y \rightarrow \infty} \frac{1}{\overline{F}^s(y)} \liminf_{n \rightarrow \infty} \mathbf{P}(W_n > y) = \frac{C}{a}. \quad (20)$$

*Remark 7.* In fact, under the conditions of Theorem 3 or 4, the condition on  $\tau_0$  (in (15)) is not required for Theorem 5.

### 3 Useful Properties

We recall some known properties of distributions. For any distribution function  $G$  on  $\mathbb{R}$  let

$$m(G) \equiv \int_{-\infty}^{\infty} t dG(t)$$

denote its mean. Further, we make the convention that, for distribution functions  $G$  and  $H$ , we write  $\overline{H}(y) \sim 0 \cdot \overline{G}(y)$  if  $\overline{H}(y) = o(\overline{G}(y))$  as  $y \rightarrow \infty$ .

**Property 1.** *Suppose that distribution functions  $G$  and  $H$  are such that  $m(G)$  is finite,  $m(H) = -h$  for some  $h > 0$ , and  $\overline{H}(y) = o(\overline{G}(y))$  as  $y \rightarrow \infty$ . Then, for any  $\varepsilon > 0$  we can find a distribution function  $H_\varepsilon$  such that  $\overline{H}(y) \leq \overline{H}_\varepsilon(y)$  for all  $y$ ,  $m(H_\varepsilon) \leq -h/2$  and  $\overline{H}_\varepsilon(y) = \varepsilon \overline{G}(y)$  for all sufficiently large  $y$ .*

**Property 2.** Suppose that distribution functions  $G$  and  $H$  are such that  $G^s$  exists, and that  $\overline{H}(y) \sim c\overline{G}(y)$  as  $y \rightarrow \infty$  for some  $c \geq 0$ . Then  $H^s$  exists and  $\overline{H}^s(y) \sim c\overline{G}^s(y)$  as  $y \rightarrow \infty$ .

**Property 3.** Suppose that a distribution function  $G$  is such that its second tail distribution  $G^s$  is long-tailed. Then

$$\overline{G}(y) = o(\overline{G}^s(y)) \quad \text{as } y \rightarrow \infty. \quad (21)$$

Further, for any  $g > 0$  and any sequence  $\{\alpha_n\}$  such that  $\alpha_n \rightarrow \alpha$  as  $n \rightarrow \infty$ ,

$$\lim_{y \rightarrow \infty} \frac{1}{\overline{G}^s(y)} \sum_{n=k}^{\infty} \alpha_n \overline{G}(y + l + ng) = \frac{\alpha}{g} \quad \text{for all } k \text{ and for all } l. \quad (22)$$

**Property 4.** Suppose that distribution functions  $G$  and  $H$  are such that  $\overline{H}(y) \sim c\overline{G}(y)$  as  $y \rightarrow \infty$  for some  $c > 0$ . Then if  $G$  is subexponential,  $H$  is subexponential, while if  $G^s$  is subexponential,  $H^s$  is subexponential and  $\overline{H}^s(y) \sim c\overline{G}^s(y)$  as  $y \rightarrow \infty$ .

**Property 5.** Let  $\xi_1, \xi_2, \dots, \xi_n$  be  $n$  mutually independent r.v.s and  $G$  a subexponential distribution such that, for  $i = 1, 2, \dots, n$ ,  $\mathbf{P}(\xi_i > y) \sim c_i \overline{G}(y)$  as  $y \rightarrow \infty$ , where  $c_1, c_2, \dots, c_n \geq 0$ . Then

$$\mathbf{P}(\xi_1 + \xi_2 + \dots + \xi_n > y) \sim (c_1 + c_2 + \dots + c_n) \overline{G}(y) \quad \text{as } y \rightarrow \infty.$$

**Property 6.** Let  $\{\xi_n\}_{n \geq 1}$  be an i.i.d. sequence of nonnegative random variables with subexponential distribution  $G$ . For any  $n$ , put

$$\alpha_n = \sup_{y \geq 0} \frac{\mathbf{P}(\xi_1 + \dots + \xi_n > y)}{\mathbf{P}(\xi_1 > y)} \equiv \sup_{y \geq 0} \frac{\overline{G}^{*n}(y)}{\overline{G}(y)}.$$

Then, for any  $u > 0$  one can choose  $k > 0$  such that  $\alpha_n \leq k(1 + u)^n$  for all  $n$ .

**Property 7** (Veraverbeke's Theorem). Let  $\{\xi_n\}_{n \geq 1}$  be an i.i.d. sequence of random variables with distribution function  $G$  and a negative mean  $-g = \mathbf{E}\xi_1 < 0$ . Suppose that the second-tail distribution  $G^s$  is subexponential. Set  $S'_n = \sum_{i=1}^n \xi_i$ , and  $M' = \max(0, \sup_n S'_n)$ . Then, as  $y \rightarrow \infty$ ,

$$\mathbf{P}(M' > y) \sim \mathbf{P}\left(\bigcup_{n \geq 1} \{\xi_n > y + ng\}\right) \sim \sum_{n \geq 1} \mathbf{P}(\xi_n > y + ng) \sim \frac{1}{g} \overline{G}^s(y). \quad (23)$$

Thus, under the conditions of Veraverbeke's Theorem, the supremum  $M'$  is large if and only if one of summands is large. The following three properties are all corollaries of Veraverbeke's Theorem. In particular Property 9 follows easily on using also Property 1 above.



**Property 8.** Under the conditions of Veraverbeke's Theorem above, for any  $\tilde{g} \in (0, g)$ ,

$$\sum_{n=1}^{\infty} \mathbf{P}(M'_n \leq y, S'_n \in (-n\tilde{g}, y], S'_{n+1} > y) = o(\overline{G}^s(y)) \quad \text{as } y \rightarrow \infty,$$

where, for each  $n$ ,  $M'_n = \max(0, \max_{1 \leq i \leq n} S'_i)$ .

**Property 9.** Let  $\{\xi_n\}_{n \geq 1}$  be an i.i.d. sequence of random variables with distribution function  $H$  and negative mean  $\mathbf{E}\xi_1 < 0$ . Suppose that  $\overline{H}(y) = o(\overline{G}(y))$ , as  $y \rightarrow \infty$ , for some distribution function  $G$  whose second-tail distribution  $G^s$  is subexponential. Set  $S'_n = \sum_{i=1}^n \xi_i$  and  $M' = \max(0, \sup_n S'_n)$ . Then

$$\mathbf{P}(M' > y) = o(\overline{G}^s(y)) \quad \text{as } y \rightarrow \infty.$$

**Property 10.** Let  $\{\xi_n\}_{n \geq 1}$  be an i.i.d. sequence of random variables with distribution function  $H$  and negative mean  $\mathbf{E}\xi_1 < 0$ . Suppose that  $\overline{H}(y) \sim c(\overline{G}(y))$ , as  $y \rightarrow \infty$ , for some  $c \geq 0$  and some distribution function  $G$  whose second-tail distribution  $G^s$  is subexponential. Let  $\tau'$  be an independent positive integer-valued random variable. Then

$$\mathbf{P}\left(\max_{1 \leq n \leq \tau'} \sum_{i=1}^n \xi_i > y\right) = o(\overline{G}^s(y)) \quad \text{as } y \rightarrow \infty.$$

## 4 Proofs

*Proof of Theorem 1.* We prove the theorem in the case where the constant  $d$  of the condition (C1) is equal to 1. The modification required for the general case is quite obvious. By the Strong Law of Large Numbers, for any  $\varepsilon \in (0, a)$ , we can choose  $R \equiv R(\varepsilon)$  such that

$$\mathbf{P}(S_n \in [-R - n(a + \varepsilon), R - n(a - \varepsilon)] \quad \text{for all } n = 0, 1, 2, \dots) \geq 1 - \varepsilon.$$

Put

$$D_n = \{S_i \in [-R - i(a + \varepsilon), R - i(a - \varepsilon)] \quad \text{for all } i = 0, 1, 2, \dots, n\}$$

and  $D \equiv D_\infty$ . Since  $D_\infty \subseteq D_n$  for all  $n$ ,  $\mathbf{P}(D_n) \geq 1 - \varepsilon$ .

Now, for all sufficiently large  $y > 0$ ,

$$\begin{aligned}
& \mathbf{P}(M > y, X_{\mu(y)} \in B) \\
&= \sum_{n=0}^{\infty} \mathbf{P}(M_n \leq y, S_{n+1} > y, X_{n+1} \in B) \\
&\geq \sum_{n=0}^{\infty} \mathbf{P}(D_n, S_{n+1} > y, X_{n+1} \in B) \\
&\geq \sum_{n=0}^{\infty} \int_B \mathbf{P}(D_n, X_{n+1} \in dx) \bar{F}_x(y + R + n(a + \varepsilon)) \\
&\geq \sum_{n=0}^{\infty} \left[ \int_B \mathbf{P}(X_{n+1} \in dx) \bar{F}_x(y + R + n(a + \varepsilon)) - \mathbf{P}(\bar{D}) L \bar{F}(y + R + n(a + \varepsilon)) \right] \\
&\geq \sum_{n=0}^{\infty} \left[ \int_B \pi(dx) \bar{F}_x(y + R + n(a + \varepsilon)) - (\mathbf{P}(\bar{D}) + \delta_{n+1}) L \bar{F}(y + R + n(a + \varepsilon)) \right],
\end{aligned}$$

where, for each  $n$ ,  $\delta_n = \sup_B |\mathbf{P}(X_n \in B) - \pi(B)|$  is the distance in total variation between the distributions of  $X_n$  and  $\pi$ . The condition (C1) implies that  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $F^s$  is long-tailed, it now follows from (4), (5) and (22) that, for all  $x$ ,

$$\lim_{y \rightarrow \infty} \frac{1}{\bar{F}^s(y)} \sum_{n=0}^{\infty} \bar{F}_x(y + R + n(a + \varepsilon)) = \frac{c(x)}{a + \varepsilon}$$

and that

$$\begin{aligned}
\limsup_{y \rightarrow \infty} \frac{1}{\bar{F}^s(y)} \sum_{n=0}^{\infty} \bar{F}_x(y + R + n(a + \varepsilon)) &\leq \lim_{y \rightarrow \infty} \frac{L}{\bar{F}^s(y)} \sum_{n=0}^{\infty} \bar{F}(y + R + n(a + \varepsilon)) \\
&= \frac{L}{a + \varepsilon}
\end{aligned} \tag{24}$$

(where the convergence to the limit above is of course independent of  $x$ ). Hence, by the Bounded Convergence Theorem,

$$\lim_{y \rightarrow \infty} \frac{1}{\bar{F}^s(y)} \sum_{n=0}^{\infty} \int_B \pi(dx) \bar{F}_x(y + R + n(a + \varepsilon)) = \frac{C(B)}{a + \varepsilon}. \tag{25}$$

Also, again from (22),

$$\lim_{y \rightarrow \infty} \frac{1}{\bar{F}^s(y)} \sum_{n=0}^{\infty} \delta_{n+1} \bar{F}(y + R + n(a + \varepsilon)) = 0.$$

Thus, again using (24),

$$\liminf_{y \rightarrow \infty} \frac{1}{\bar{F}^s(y)} \mathbf{P}(M > y, X_{\mu(y)} \in B) \geq \frac{C(B) - L \mathbf{P}(\bar{D})}{a + \varepsilon}.$$

Now let  $\varepsilon \rightarrow 0$  to obtain the required result. □

We now give two lemmas which are required for the remaining results.

**Lemma 1.** *Suppose that the conditions of Theorem 1 hold and that*

$$\limsup_{y \rightarrow \infty} \frac{\mathbf{P}(M > y)}{\overline{F}^s(y)} \leq \frac{C}{a}. \quad (26)$$

*Then the conclusion (11) follows.*

*Proof.* From (26), for any  $B \in \mathcal{X}$ ,

$$\begin{aligned} \frac{C}{a} &\geq \limsup_{y \rightarrow \infty} \frac{\mathbf{P}(M > y)}{\overline{F}^s(y)} \\ &= \limsup_{y \rightarrow \infty} \left( \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \frac{\mathbf{P}(M > y, X_{\mu(y)} \in \overline{B})}{\overline{F}^s(y)} \right) \\ &\geq \limsup_{y \rightarrow \infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \liminf_{y \rightarrow \infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in \overline{B})}{\overline{F}^s(y)} \\ &\geq \limsup_{y \rightarrow \infty} \frac{\mathbf{P}(M > y, X_{\mu(y)} \in B)}{\overline{F}^s(y)} + \frac{C(\overline{B})}{a}, \end{aligned}$$

where the last inequality follows by Theorem 1. Since  $C = C(B) + C(\overline{B})$ , the conclusion (11) follows as required.  $\square$

In each of the proofs of Theorems 2, 3 and 4 we show that, for all  $\varepsilon$  satisfying  $0 < \varepsilon < a$ , there exists  $R > 0$  (depending on  $\varepsilon$ ) such that, if, for each  $n = 1, 2, \dots$ ,

$$D'_n \equiv \{S_j \leq R - j(a - \varepsilon) \text{ for all } j = 1, \dots, n-1; S_{n+i} - S_n \leq R \text{ for all } i = 1, 2, \dots\}, \quad (27)$$

then  $\mathbf{P}(D'_n) > 1 - \varepsilon$  for all  $n$ . In each case we then require Lemma 2 below to complete the proof.

**Lemma 2.** *Suppose that  $F^s$  is subexponential, that there exist a sequence of i.i.d. random variables  $\{\psi_n\}_{n \geq 1}$  and a constant  $L_1$  such that  $\mathbf{E}\psi_1 < 0$  and*

$$\mathbf{P}(\psi_1 > y) \leq L_1 \overline{F}(y) \quad (28)$$

*for all  $y \geq 0$ , and that  $\psi_n$  is independent of  $D'_n$  for all  $n \geq 1$ . Suppose further that the condition (C1) is satisfied and that*

$$\mathbf{P}(M > y) \leq \mathbf{P}(M > y, M^\psi > y) + o(\overline{F}^s(y)) \quad \text{as } y \rightarrow \infty, \quad (29)$$

*where  $M^\psi = \max(0, \sup_n \sum_{i=1}^n \psi_i)$ . Then the conclusion (11) follows.*

*Proof.* As in the proof of Theorem 1, we assume that the constant  $d$  of the condition (C1) is equal to 1. We may further assume, without loss of generality, that the condition (28) is satisfied with equality for all sufficiently large  $y$ . (If this is not the case we can use Property 1 of Section 3 to replace  $\{\psi_n\}_{n \geq 1}$  with i.i.d. sequence  $\{\tilde{\psi}_n\}_{n \geq 1}$  satisfying all the conditions of the lemma and with also the required equality in (28).) It follows that the common distribution of the random variables  $\psi_n$  has a second-tail distribution which is subexponential. Thus, if  $g = -\mathbf{E}(\psi_1)$  (so  $g > 0$ ), it follows from the conditions of the lemma and Veraverbeke's Theorem that

$$\begin{aligned} \mathbf{P}(M > y) &\leq \mathbf{P}(M > y, M^\psi > y) + o(\bar{F}^s(y)) \\ &= \sum_{n=1}^{\infty} \mathbf{P}(M > y, \psi_n > y + ng) + o(\bar{F}^s(y)) \\ &\leq \Sigma_1 + \Sigma_2 + o(\bar{F}^s(y)), \end{aligned} \tag{30}$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} \mathbf{P}(D'_n, M > y), \quad \Sigma_2 = \sum_{n=1}^{\infty} \mathbf{P}(\bar{D}'_n, M > y, \psi_n > y + ng).$$

Since, for each  $n$ ,  $\psi_n$  is independent of  $D'_n$ , we have, using (22),

$$\Sigma_2 \leq \sum_n \mathbf{P}(\bar{D}'_n) \mathbf{P}(\psi_n > y + ng) \leq (1 + o(1)) \frac{\varepsilon L_1}{g} \bar{F}^s(y) \quad \text{as } y \rightarrow \infty. \tag{31}$$

We now consider  $\Sigma_1$ . Take  $y > R$ . For any  $n$ , the event

$$V_n \equiv D'_n \cap \{\xi_n^{X_n} \leq y - 2R + (n-1)(a - \varepsilon)\} \subseteq \{M \leq y\}. \tag{32}$$

To see this, note that, on the set  $V_n$ ,  $S_j \leq R - j(a - \varepsilon)$  for all  $j < n$ ,

$$S_n = S_{n-1} + \xi_n^{X_n} \leq y - R$$

and, for  $i = 1, 2, \dots$ ,

$$S_{n+i} = S_n + (S_{n+i} - S_n) \leq y - R + R = y.$$

Thus, from (32),

$$\begin{aligned} \Sigma_1 &\leq \sum_n \mathbf{P}(\xi_n^{X_n} > y - 2R + (n-1)(a - \varepsilon)) \\ &= \sum_n \int_{\mathcal{X}} \mathbf{P}(X_n \in dx) \bar{F}_x(y - 2R + (n-1)(a - \varepsilon)) \\ &\leq \sum_n \int_{\mathcal{X}} \pi(dx) \bar{F}_x(y - 2R + (n-1)(a - \varepsilon)) + L \sum_n \delta_n \bar{F}(y - 2R + (n-1)(a - \varepsilon)), \end{aligned}$$

where, as in the the proof of Theorem 1,  $\delta_n$  is the distance in total variation between the distributions of  $X_n$  and  $\pi$ , and so tends to 0 as  $n \rightarrow \infty$ . Exactly as in that proof, it now follows from (22) and the Bounded Convergence Theorem that

$$\limsup_{y \rightarrow \infty} \frac{\Sigma_1}{\overline{F}^s(y)} \leq \frac{C}{a - \varepsilon}.$$

It now follows, on recalling (30) and (31) and letting  $\varepsilon \rightarrow 0$ , that the condition (26) of Lemma 1 is satisfied. The required conclusion (11) now follows from that lemma.  $\square$

*Proof of Theorem 2.* It follows from the condition (5) that, without loss of generality, we can assume that  $\overline{G}(y) \leq L\overline{F}(y)$  for all  $y$ . Let  $\{\alpha_n\}_{n \geq 1}$  be an i.i.d. sequence of random variables uniformly distributed on  $(0, 1)$  and independent of  $X = \{X_n\}$ . Construct the required family of random variables  $\{\xi_n^x\}_{n \geq 1}$  by defining, for each  $n$ ,  $\xi_n^x = F_x^{-1}(\alpha_n)$ ; for each  $n$  define also  $\psi_n = G^{-1}(\alpha_n)$ . Here, for any distribution function  $H$ , the quantile function  $H^{-1}$  is given by

$$H^{-1}(t) = \sup\{z : H(z) \leq t\}.$$

Note that the pairs  $(\xi_n^x, \psi_n)$ ,  $n \geq 1$ , are independent in  $n$ , that the sequence  $\{\psi_n\}_{n \geq 1}$  is i.i.d. with  $\mathbf{E}\psi_1 < 0$  (from (10)) and distribution function  $G$ , and that

$$\xi_n^x \leq \psi_n \quad \text{a.s.} \quad (33)$$

Put  $S_n^\psi = \sum_{j=1}^n \psi_j$  and

$$M^\psi = \max(0, \sup_n S_n^\psi).$$

From the SLLN for  $\{\psi_n\}$  and from the condition (C2), for any  $\varepsilon > 0$ , there exists  $R > 0$  such that, for any  $n = 1, 2, \dots$ ,

$$\mathbf{P}(S_j < R - j(a - \varepsilon), j = 1, 2, \dots, n-1; S_{n+i}^\psi - S_n^\psi < R, i = 1, 2, \dots) > 1 - \varepsilon.$$

Hence, from (33),  $\mathbf{P}(D'_n) > 1 - \varepsilon$  for all  $n$ , where each  $D'_n$  is as given by (27). Also from (33),

$$\mathbf{P}(M > y) = \mathbf{P}(M > y, M^\psi > y).$$

It is now easy to check that all the conditions of Lemma 2 are satisfied, with each  $\psi_n$  and  $D'_n$  as given here, and the required result now follows from that lemma.  $\square$

The following further two lemmas are also required in each of the proofs of Theorems 3 and 4 (where in each case  $X$  is regenerative).

**Lemma 3.** *Suppose that  $X$  is regenerative with  $\mathbf{E}\tau < \infty$  and also that  $F^s$  is subexponential. Let  $\{\{\eta_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  be a sequence of families of random variables such that these families are independent and identically distributed in  $n$  and are further independent of  $X$ . Suppose further that there exists a constant  $b > 0$  satisfying the condition (15) and such that*

$$\eta_1^x \leq b \quad \text{a.s.} \quad \text{for all } x, \quad (34)$$

and that

$$\int_{\mathcal{X}} \mathbf{E}\eta_1^x \pi(dx) < 0. \quad (35)$$

Define

$$M^\eta = \max \left( 0, \sup_n \sum_{i=1}^n \eta_i^{X_i} \right).$$

Then

$$\mathbf{P}(M^\eta > y) = o(\overline{F}^s(y)) \quad \text{as } y \rightarrow \infty.$$

*Proof.* Define

$$\beta_n = \sum_{i=T_{n-1}+1}^{T_n} \eta_i^{X_i}, \quad n \geq 1.$$

Observe that  $\{\beta_n\}_{n \geq 1}$  is an i.i.d. sequence with, from (34) and (35),

$$\mathbf{E}\beta_1 < 0, \quad \beta_n \leq b\tau_n, \quad n \geq 1.$$

Since also  $X$  is regenerative with  $\mathbf{E}(\tau) < \infty$ , we can choose  $K > 0$  sufficiently large that if

$$\gamma_n = \max(\beta_n, b\tau_n - K), \quad n \geq 1, \quad (36)$$

then  $\{\gamma_n\}_{n \geq 1}$  is an i.i.d. sequence with

$$\mathbf{E}\gamma_1 < 0, \quad \gamma_n \leq b\tau_n, \quad n \geq 1. \quad (37)$$

Define also

$$M^\gamma = \max \left( 0, \sup_{n \geq 1} \sum_{i=1}^n \gamma_i \right).$$

It follows from (37), the assumed condition (15) (for  $b$  as given) and the extension of Veraverbeke's Theorem given by Property 9 of Section 3, that

$$\mathbf{P}(M^\gamma > y) = o(\overline{F}^s(y)), \quad \text{as } y \rightarrow \infty. \quad (38)$$

Now

$$\begin{aligned} M^\eta &\leq b\tau_0 + \sup(b\tau_1, \beta_1 + b\tau_2, \beta_1 + \beta_2 + b\tau_3, \dots) \\ &\leq b\tau_0 + K + \sup(\gamma_1, \gamma_1 + \gamma_2, \gamma_1 + \gamma_2 + \gamma_3, \dots) \\ &\leq b\tau_0 + K + M^\gamma, \end{aligned}$$

where the second inequality above follows from (36). Further  $\tau_0$  and  $M^\gamma$  are independent. The required result now follows from (38), the assumed condition (15) and Property 5 of Section 3.  $\square$

The following lemma combines the results of Lemmas 2 and 3 to provide a set of conditions for the regenerative case under which there follows the desired conclusion (11) of both Theorems 3 and 4.

**Lemma 4.** *Suppose that  $X$  is regenerative with  $\mathbf{E}\tau < \infty$  and also that  $F^s$  is subexponential. Suppose also that there exist a sequence of i.i.d. random variables  $\{\psi_n\}_{n \geq 1}$  and a constant  $L_1$  satisfying the conditions of Lemma 2, i.e. that*

$$\mathbf{E}(\psi_1) < 0, \quad \mathbf{P}(\psi_1 > y) \leq L_1 \bar{F}(y) \quad \text{for all } y \geq 0, \quad (39)$$

and that

$$\psi_n \text{ is independent of } D'_n \text{ for all } n \geq 1 \quad (40)$$

(where  $D'_n$  is as given by (27)). Suppose further that there exists a sequence of families of random variables  $\{\{\eta_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  and a constant  $b > 0$  satisfying all the conditions of Lemma 3, and that

$$\xi_n^x \leq \psi_n + \eta_n^x, \quad x \in \mathcal{X}, \quad n \geq 1. \quad (41)$$

Again define

$$M^\psi = \max \left( 0, \sup_n \sum_{i=1}^n \psi_i \right), \quad M^\eta = \max \left( 0, \sup_n \sum_{i=1}^n \eta_i^{X_i} \right).$$

Finally, suppose that  $M^\psi$  and  $M^\eta$  are independent. Then the conclusion (11) follows.

*Proof.* As in the proof of Lemma 2 we may assume, without loss of generality, that  $\mathbf{P}(\psi_1 > y) = L_1 \bar{F}(y)$  for all sufficiently large  $y$ . It then follows from the conditions on the sequence  $\{\psi_n\}$  and Veraverbeke's Theorem that

$$\mathbf{P}(M^\psi > y) \sim L_1 \bar{F}^s(y), \quad \text{as } y \rightarrow \infty, \quad (42)$$

while it follows from Lemma 3 that

$$\mathbf{P}(M^\eta > y) = o(\bar{F}^s(y)), \quad \text{as } y \rightarrow \infty. \quad (43)$$

From the condition (41) we have that

$$M \leq M^\psi + M^\eta. \quad (44)$$

Since also  $M^\psi$  and  $M^\eta$  are independent, it now follows from (42), (43), (44) and Property 5 of Section 3 that

$$\mathbf{P}(M > y) = \mathbf{P}(M > y, M^\psi > y) + o(\bar{F}^s(y)) \quad \text{as } y \rightarrow \infty. \quad (45)$$

Finally, since  $X$  is regenerative, the condition (C1), and so now all the conditions of Lemma 2, are satisfied and so the required conclusion (11) again follows from that lemma.  $\square$

*Proof of Theorem 3.* We construct the sequences  $\{\psi_n\}_{n \geq 1}$  and  $\{\{\eta_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  and the constants  $L_1$  and  $b$  such that all the conditions of Lemma 4 are satisfied.

It follows from (5) that we can find a distribution function  $G$  on  $\mathbb{R}$  such that

$$\overline{F}_x(y) \leq \overline{G}(y) \leq L\overline{F}(y), \quad (46)$$

for all  $y$  and for all  $x \in \mathcal{X}$ . As in the proof of Theorem 2, let  $\{\alpha_n\}_{n \geq 1}$  be an i.i.d. sequence of random variables uniformly distributed on  $(0, 1)$  and independent of  $X = \{X_n\}$ . Again construct the required family of random variables  $\{\xi_n^x\}_{n \geq 1}$  by defining, for each  $n$ ,  $\xi_n^x = F_x^{-1}(\alpha_n)$ ; for each  $n$  define also  $\zeta_n = G^{-1}(\alpha_n)$ . Then the pairs  $(\xi_n^x, \zeta_n)$ ,  $n \geq 1$ , are independent in  $n$ , the sequence  $\{\zeta_n\}_{n \geq 1}$  is i.i.d., and

$$\xi_n^{X_n} \leq \zeta_n \quad \text{a.s.,} \quad \text{for all } n. \quad (47)$$

For  $y_0 > 0$ , define

$$u(y_0) = \mathbf{E}[\mathbf{I}(\zeta_1 > y_0)\zeta_1], \quad v(y_0) = - \int_{\mathcal{X}} \mathbf{E}[\mathbf{I}(\zeta_1 \leq y_0)\xi_1^x]\pi(dx).$$

Observe that  $u(y_0) \rightarrow 0$  as  $y_0 \rightarrow \infty$  and, by the conditions (13) and (14),  $v(y_0) \rightarrow a$  as  $y_0 \rightarrow \infty$ . Choose  $y_0$  sufficiently large and  $K > 0$  such that

$$u(y_0) < \mathbf{P}(\zeta_1 > y_0)K < v(y_0). \quad (48)$$

We define the required i.i.d. sequence  $\{\psi_n\}_{n \geq 1}$  by

$$\psi_n = \mathbf{I}(\zeta_n > y_0)(\zeta_n - K).$$

It follows from the construction of this sequence, and in particular from (46), (48) and the definition of  $u(y_0)$ , that it satisfies the conditions (39) and (40) of Lemma 4 with  $L_1 = L$ . Define also, for each  $n$  and for each  $x$ ,

$$\eta_n^x = \mathbf{I}(\zeta_n \leq y_0)\xi_n^x + \mathbf{I}(\zeta_n > y_0)K,$$

The random variables  $\eta_n^x$  are bounded above by  $b = \max(y_0, K)$ . Further, by (48) and the definition of  $v(y_0)$ ,

$$\int_{\mathcal{X}} \mathbf{E}\eta_1^x \pi(dx) < 0.$$

It now follows, using also the condition (15) of the theorem, that the sequence  $\{\{\eta_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  and  $b$  as given above satisfy the conditions of Lemma 3, and so also of Lemma 4.

The condition (41) follows on observing that, from (47), for all  $x$  and for all  $n$ ,

$$\begin{aligned} \xi_n^x &= \mathbf{I}(\zeta_n > y_0)(\xi_n^x - K) + \mathbf{I}(\zeta_n \leq y_0)\xi_n^x + \mathbf{I}(\zeta_n > y_0)K \\ &\leq \psi_n + \eta_n^x. \end{aligned}$$

Finally, it is not difficult to see that the random variables  $M^\psi$  and  $M^\eta$  (defined as in the statement of Lemma 4) are independent (although the sequences  $\{\psi_n\}$  and  $\{\eta_n^{X_n}\}$  of which they are the maxima are *not* independent!). The required conclusion (11) now follows from Lemma 4.  $\square$



*Proof of Theorem 4.* We again use Lemma 4. It follows from the conditions of the theorem that we may take  $b$  such that

$$\mathbf{E}\zeta < b < \mathbf{E}_\pi b^X \quad (49)$$

and satisfying (15). It follows also from (18) that there exists  $L_1 > 0$  such that

$$\mathbf{P}(\zeta > y) \leq L_1 \bar{F}(y) \quad (50)$$

for all  $y$ . Further, we may define random variables  $\{\xi^x\}_{x \in \mathcal{X}}$ ,  $\zeta$ , and  $\{b^x\}_{x \in \mathcal{X}}$  in such a way that  $\zeta$  and the family  $\{b^x\}_{x \in \mathcal{X}}$  are independent and, for all  $x$ ,

$$\xi^x \leq \zeta - b^x \quad \text{a.s.} \quad (51)$$

For  $n = 1, 2, \dots$ , let  $\{\xi_n^x, b_n^x, \zeta_n\}$  be i.i.d. copies of  $\{\xi^x, b^x, \zeta\}$ , such that these sequences are jointly independent of the process  $X$ . Define, for all  $n$ ,

$$\psi_n = \zeta_n - b, \quad \eta_n^x = b - b_n^x, \quad x \in \mathcal{X}. \quad (52)$$

Then it is easy to check, from (49)–(52) and the condition (15) and the independence assumption of the theorem, that all the conditions of Lemma 4 are satisfied. The sequences  $\{\psi_n\}_{n \geq 1}$  and  $\{\{\eta_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  and the constants  $L_1$  and  $b$  of that lemma are as given here. We thus have the required result.  $\square$

*Proof of Theorem 5.* We again give the proof in the case  $d = 1$ . It follows straightforwardly from the regenerative structure of  $X$ , the condition  $\mathbf{E}\tau < \infty$ , and the condition (C2) that the random vectors

$$\begin{aligned} Y_0 &= \{\tau_0; W_1, \dots, W_{T_0}\}, \\ Y_n &= \{\tau_n; W_{T_{n-1}+1}, \dots, W_{T_n}\}, \quad n \geq 1, \end{aligned}$$

form a Harris ergodic Markov chain (see, for example, [11]). Then, since  $d = 1$ , it is again straightforward that  $W_n$  converges in the total variation norm to a distribution on  $\mathbb{R}_+$  which is independent of that of  $Y_0$ . Now let  $\tilde{X} = \{\tilde{X}_n\}_{-\infty < n < \infty}$  be the corresponding stationary version of the process  $X$  indexed over the entire set of integers, and similarly extend the i.i.d. sequence of families  $\{\{\xi_n^x\}_{x \in \mathcal{X}}\}_{n \geq 1}$  to  $\{\{\xi_n^x\}_{x \in \mathcal{X}}\}_{-\infty < n < \infty}$ . Let  $\{\tilde{W}_n\}_{n \geq 0}$  (with  $\tilde{W}_0 \equiv 0$  as usual) be the corresponding version of the process  $\{W_n\}_{n \geq 0}$ . It follows from the recursion (3) that

$$\tilde{W}_n = \max \left( 0, \xi_n^{\tilde{X}_n}, \xi_n^{\tilde{X}_n} + \xi_{n-1}^{\tilde{X}_{n-1}}, \dots, \xi_n^{\tilde{X}_n} + \dots + \xi_1^{\tilde{X}_1} \right)$$

which, by stationarity, has the same distribution as

$$\max \left( 0, \xi_{-1}^{\tilde{X}_{-1}}, \xi_{-1}^{\tilde{X}_{-1}} + \xi_{-2}^{\tilde{X}_{-2}}, \dots, \xi_{-1}^{\tilde{X}_{-1}} + \dots + \xi_{-n}^{\tilde{X}_{-n}} \right).$$

Thus, for any  $y$ ,  $\lim_{n \rightarrow \infty} \mathbf{P}(W_n > y)$  and  $\lim_{n \rightarrow \infty} \mathbf{P}(\tilde{W}_n > y)$  both exist and are equal to  $\mathbf{P}(M^- > y)$  where

$$M^- = \sup \left( 0, \xi_{-1}^{\tilde{X}_{-1}}, \xi_{-1}^{\tilde{X}_{-1}} + \xi_{-2}^{\tilde{X}_{-2}}, \dots \right).$$

The required result now follows from the application of Theorem 2, 3 or 4 as appropriate, in each case with  $B = \mathcal{X}$ , to the time-reversed version of the stationary process  $\{\tilde{X}_n, \xi_n^{\tilde{X}_n}\}$ . However, under the conditions of Theorem 3 or Theorem 4, we must also verify the required condition on  $\tau_0^-$ , defined to be the time of the first regeneration at or after time 0 in the reversed process  $X^- = \{X_n^-\}_{n \geq 0}$  given by  $X_n^- = \tilde{X}_{-n}$ . Standard renewal theory shows that the distribution of  $\tau_0^-$  is given by

$$\mathbf{P}(\tau_0^- \geq n) = \frac{1}{\mathbf{E}(\tau)} \sum_{k=n+1}^{\infty} \mathbf{P}(\tau \geq k), \quad n = 0, 1, \dots$$

An easy calculation, analogous to that of the derivation of Property 2 of Section 3, now gives that, if  $b > 0$  is such that  $\mathbf{P}(b\tau > y) = o(\bar{F}(y))$  as  $y \rightarrow \infty$ , then  $\mathbf{P}(b\tau_0^- > y) = o(\bar{F}^s(y))$  as  $y \rightarrow \infty$ . Thus, in each case, the required condition on  $\tau_0^-$  follows from the assumed condition on  $\tau$ .

The modifications for the case of general  $d$  are again routine.  $\square$

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